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GRAPH DECOMPOSITION WITH CONSTRAINTS ON THE MINIMUM DEGREE

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Let G be a finite simple graph on n vertices with minimum degree $\delta(G) \geqslant \delta$ ($n \equiv \delta \pmod 2$). Let $\max(k, G)$ denote the set of all k-subsets $A \subseteq V(G)$ such that the number of edges in the induced subgraph $\langle A \rangle$ is a maximum. We prove that for some $i \in \{0, 1, 2, \ldots, \lfloor \frac{1}{2}\delta \rfloor\}$ there exists a partition (X, Y) of $V(G)$ such that (i) $(X) = \lceil \frac{1}{2}n \rceil + i$, $|Y| = \lfloor \frac{1}{2}n \rfloor - i$; (ii) $\delta(X) \geqslant \lceil \frac{1}{2}\delta \rceil + i$, $\delta(Y) \geqslant \lfloor \frac{1}{2}\delta \rfloor - i$; (iii) $X \in \max(\lceil \frac{1}{2}n \rceil + i, G)$ or $Y \in \max(\lfloor \frac{1}{2}n \rfloor - i, G)$. Analogous edge density constraints, rather than constraints on the minimum degree of G, guaranteeing such a partition are also discussed.

Introduction

A classical argument due to Erdös shows that every finite graphs G with minimum degree δ contains an induced bipartite graph H with $\delta(H) \geqslant \lceil \frac{1}{2}\delta \rceil$. Jackson [8] has proved that if $\delta \geqslant 2$ then there exists a *balanced* induced bipartite subgraph H with $\delta(H) \geqslant 1$. Other papers in this general area include [1, 3, 6, 7, 10, 11, 12, 13, 17 and 18]. Thomassen [16] has proved that every graph G such that $\delta(G) \geqslant 12k$ contains a partition (A, B) of $V(G)$ such that $\delta(A) \geqslant k$ and $\delta(B) \geqslant k$. There is obviously no analogue of the Jackson result where the relative cardinalities of A and B are given. We prove in Theorem 1 a result which is, at least superficially, related to this question. Let $\max(k, G)$ denote the set of all k-subsets $A \subseteq V(G)$ such that $q(A)$—the number of edges in the induced subgraph $\langle A \rangle$—is a maximum subject to these conditions. Let $\max(G) = \bigcup_{k=1}^{n} \max(k, G)$, ($n = |V(G)|$). So $X \in \max(G)$ if and only if amongst all subsets of $V(G)$ of cardinality $|X|$, $q(X)$ is a maximum. Elements of $\max(G)$ are called *dense sets*. Theorem 1 states that if $\delta(G) \geqslant \delta$, then every graph G on n vertices contains for some $i \in \{0, 1, 2, \ldots, \lfloor \frac{1}{2}\delta \rfloor\}$ a partition (X, Y) of $V(G)$ such that (i) $|X| = \lceil \frac{1}{2}n \rceil + i$, $|Y| = \lfloor \frac{1}{2}n \rfloor - i$; (ii) $\delta(X) \geqslant \lceil \frac{1}{2}\delta \rceil + i$, $\delta(Y) \geqslant \lfloor \frac{1}{2}\delta \rfloor - i$; (iii) either X or Y is dense. Theorem 1 requires that if n is even then δ is even. Theorem 2 gives an analogous result in the exceptional case. In Section 3 we discuss (see Conjecture 1) the more difficult question of determining the number of edges in G which guarantees the existence of such a decomposition of G without any constraints on the minimum degree of G. Finally in Section 4 we speculate on some possible generalizations involving for example the connectivity of G.

1. Theorems 1 and 2

All graphs are both simple and finite. In general our notation will follow [2]. Let G be a graph with n vertices. The minimum degree of G is denoted by $\delta(G)$. Suppose $X \subseteq V(G)$. Then the induced subgraph of G with vertex set X is denoted by $\langle X \rangle$. We write $\delta(X)$ rather than $\delta(\langle X \rangle)$ and we use similar abbreviations without further reference.

Suppose i and δ are integers such that $\lfloor \frac{1}{2}\delta \rfloor \geq i \geq 0$, $n-1 \geq \delta \geq 0$. A partition (X, Y) of $V(G)$ is an (i, δ)-*partition* if:

(1) $|X| = \lceil \frac{1}{2}n \rceil + i$, $|Y| = \lfloor \frac{1}{2}n \rfloor - i$,

(2) $\delta(X) \geq \lceil \frac{1}{2}\delta \rceil + i$, $\delta(Y) \geq \lfloor \frac{1}{2}\delta \rfloor - i$,

(3) either X or Y is dense.

Let

$$\mathscr{C}(n, \delta, i) = \{G : |V(G)| = n,\ G \text{ contains an } (i, \delta)\text{-partition}\}$$

and let

$$\mathscr{C}(n, \delta) = \bigcup_{i=0}^{s} \mathscr{C}(n, \delta, i),$$

where $s = \lfloor \frac{1}{2}\delta \rfloor$.

A *weak* (i, δ)-*partition* (X, Y) of $V(G)$ satisfies conditions (1) and (2), i.e., we no longer insist that either X or Y is dense. For weak partitions we use the notation $\omega(n, \delta, i)$ and $\omega(n, \delta)$ rather than $\mathscr{C}(n, \delta, i)$ and $\mathscr{C}(n, \delta)$.

Theorem 1. *Let G be a graph with n vertices. Suppose δ is an integer such that $n-1 \geq \delta \geq 0$ and suppose that if n is even, then δ is even. Suppose $\delta(G) \geq \delta$. Then $G \in \mathscr{C}(n, \delta)$.*

Remarks (Sharpness)

(1) When n is even and δ odd G does not necessarily belong to $\mathscr{C}(n, \delta)$. For example for any non-negative integer s, $K_{2s+1,2s+1} \notin \mathscr{C}(4s+2, 2s+1)$. A more complicated example of a graph G where $G \notin \mathscr{C}(n, \delta)$ is shown in Fig. 1. The graph G has $p(G) = 4s+2$ ($p(G)$ denotes $|V(G)|$) and $\delta(G) = 2s+1$ and may be described as follows: $V(G) = A_1 \cup A_2 \cup B_1 \cup B_2$, where A_1, A_2, B_1 and B_2 are pairwise disjoint sets such that $|A_i| = s$ and $|B_i| = s+1$ $(i = 1, 2)$. Furthermore

$$q(G) = 2\binom{s}{2} + 2s(s+1) + (s+1)^2 + s,$$

where G contains a perfect matching of A_1 into A_2 and

$$q(A_i) = \binom{s}{2}, \qquad q(A_i, B_i) = s(s+1) \quad (i = 1, 2),$$

$$q(B_1, B_2) = (s+1)^2, \qquad q(A_1, A_2) = s.$$

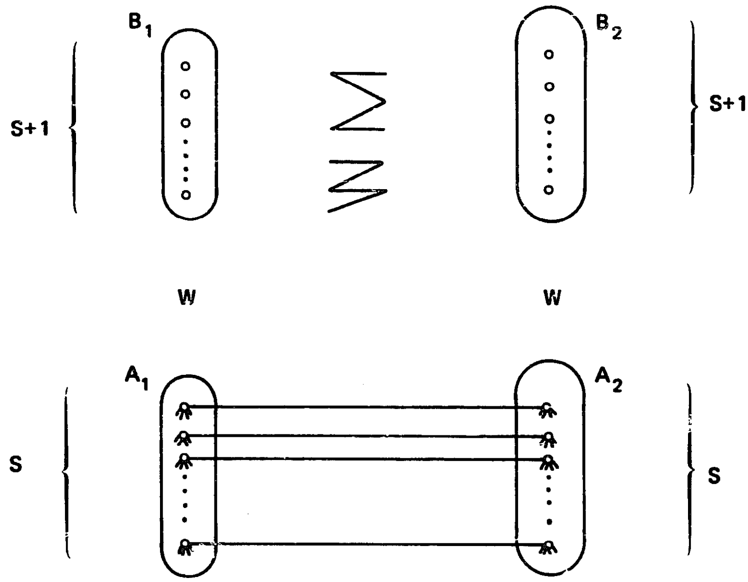

Fig. 1.

(Here, if $X, Y \subseteq V(G)$, we write $q(X, Y) = |\{xy \in E(G): x \in X, y \in Y\}|$. For $q(X, X)$ we write $q(X)$ and we use similar abbreviations without further reference).

In Theorem 2 we give a weaker result when n is even and δ is odd.

(2) The theorem is also best possible in the following sense. There exists a graph G on n vertices such that if δ is any integer satisfying $n-2 \geq \delta \geq 1$, $n \equiv \delta \pmod 2$ and $\delta(G) \geq \delta - 1$, then $G \notin \mathscr{C}(n, \delta)$. For example when

$$G = K_{\delta-1} + (\bar{K}_{\lfloor\frac{1}{2}n\rfloor - \lfloor\frac{1}{2}\delta\rfloor + 1} \cup K_{\lceil\frac{1}{2}n\rceil - \lceil\frac{1}{2}\delta\rceil}).$$

(3) The theorem is also best possible in the following sense. Suppose s and t are integers such that $1 \leq s \leq t < 2s$ and $s \equiv t \pmod 2$. Let $G = K_s \cup K_t$ and suppose $i \in \{0, 1, \ldots, \lfloor\frac{1}{2}\delta\rfloor\}$. Then $G \in \mathscr{C}(s+t, s-1, i)$ if and only if $i = \frac{1}{2}(t-s)$. On the other hand if $1 \leq s \leq t$ and t is even than $K_{s,t} \in \mathscr{C}(s+t, s, i)$ for $i \in \{0, 1, \ldots, \lfloor\frac{1}{2}(t-s)\rfloor\}$.

The next theorem deals with the case n even and δ odd. The proof of Theorem 2 is similar to the proof of Theorem 1 and we do not include the details.

Theorem 2. *Let G be a graph with n vertices. Suppose δ is an integer such that $n-1 \geq \delta \geq 1$ and suppose n is even and δ is odd. Then either*:

(1) *there exists a partition (X, Y) of $V(G)$ such that*: (i) $|X| = |Y| = \frac{1}{2}n$; (ii) $\delta(X) \geq \lceil\frac{1}{2}\delta\rceil$, $\delta(Y) \geq \lceil\frac{1}{2}\delta\rceil$; (iii) *either X or Y is dense, or*

(2) *for some $i \in \{1, 2, \ldots, \lceil\frac{1}{2}\delta\rceil\}$ there exists a partition (X_i, Y_i) of $V(G)$ such that*: (i) $|X_i| = \frac{1}{2}n + i$, $|Y_i| = \frac{1}{2}n - i$; (ii) $\delta(X_i) \geq \lceil\frac{1}{2}\delta\rceil + i - 1$, $\delta(Y_i) \geq \lfloor\frac{1}{2}\delta\rfloor - i + 1$; (iii) *$X$ or Y is dense.*

2. Proof of Theorem 1

Lemma. *Suppose k is an integer such that $1 \leqslant k \leqslant n-1$.*

(i) *Suppose $X_0 \in \max(k, G)$. Write $Y_0 = V(G)\backslash X_0$. Then, if $X_1 \in \max(k+1, G)$, then $\delta(X_1) \geqslant \delta - \delta(Y_0)$.*

(ii) *Suppose $Y_0 \in \max(k+1, G)$. Then, if $Y_1 \in \max(k, G)$ and $X_1 = V(G)\backslash Y_1$, then $\delta(X_1) \geqslant \delta - \delta(Y_0)$.*

Proof. (i) Suppose $X_0 \in \max(k, G)$. Write $Y_0 = V(G)\backslash X_0$. Then, since $k \leqslant n-1$, $|Y_0| \geqslant 1$. Choose $y_0 \in Y_0$ so that $q(y_0, Y_0) = \delta(Y_0)$. Write $X_1^* = X_0 \cup \{y_0\}$. Let $X_1 \in \max(k+1, G)$. Then, since $|X_1^*| = k+1$,

$$q(X_1) \geqslant q(X_1^*) = q(X_0) + q(y_0, X_0). \tag{1}$$

Let $x_1 \in X_1$. Write $X_0^* = X_1 - \{x_1\}$. Then $|X_0^*| = k$ and since $X_0 \in \max(k, G)$,

$$q(X_0) \geqslant q(X_0^*) = q(X_1) - q(x_1, X_1). \tag{2}$$

From (1) and (2)

$$\begin{aligned} q(x_1, X_1) &\geqslant q(y_0, X_0) = \deg(y_0) - q(y_0, Y_0) \\ &\geqslant \delta - \delta(Y_0), \end{aligned} \tag{3}$$

where $\deg(y_0)$ is the degree of y_0 in G. Hence, from (3),

$$\delta(X_1) \geqslant \delta - \delta(Y_0) \quad (X_1 \in \max(k+1, G)).$$

(ii) Suppose $Y_0 \in \max(k+1, G)$. Then $Y_0 \neq \emptyset$. Choose $y_0 \in Y_0$ such that $q(y_0, Y_0) = \delta(Y_0)$. Write $Y_2 = Y_0 - \{y_0\}$. Let $Y_1 \in \max(k, G)$ and write $X_1 = V(G)\backslash Y_1$. Since $|Y_2| = k$ and $Y_1 \in \max(k, G)$,

$$q(Y_1) \geqslant q(Y_2) = q(Y_0) - q(y_0, Y_0). \tag{4}$$

Let $x_1 \in X_1$. Write $Y_3 = Y_1 \cup \{x_1\}$. Then since $|Y_3| = k+1$ and $Y_0 \in \max(k+1, G)$,

$$q(Y_0) \geqslant q(Y_3) = q(Y_1) + q(x_1, Y_1). \tag{5}$$

From (4) and (5)

$$\begin{aligned} 0 &\geqslant q(x_1, Y_1) - q(y_0, Y_0) \\ &= \deg(x_1) - q(x_1, X_1) - q(y_0, Y_0), \end{aligned} \tag{6}$$

i.e.,

$$\begin{aligned} q(x_1, X_1) &\geqslant \deg(x_1) - q(y_0, Y_0) \\ &\geqslant \delta - \delta(Y_0). \end{aligned} \tag{7}$$

Hence, from (7),

$$\delta(X_1) \geqslant \delta - \delta(Y_0) \quad (Y_1 \in \max(k, G)). \quad \square$$

Proof of Theorem 1. Assume that $G \notin \mathscr{C}(n, \delta)$. Choose $X \in \max(\lceil \frac{1}{2}n \rceil, G)$.

Case 1. Suppose X can be chosen so that $\delta(X) \geq \lceil \frac{1}{2}\delta \rceil$. Suppose $i \in \{0, 1, \ldots, \lfloor \frac{1}{2}\delta \rfloor\}$. A partition (X_i, Y_i) of $V(G)$ is defined recursively as follows:

(i) $X_0 = X$, $Y_0 = V(G) \backslash X_0$,

(ii) using Lemma (i), choose (X_{i+1}, Y_{i+1}) so that

$$X_{i+1} \in \max(|X_i| + 1, G), \qquad Y_{i+1} = V(G) \backslash X_{i+1} \tag{8}$$

and

$$\delta(X_{i+1}) \geq \delta - \delta(Y_i). \tag{9}$$

From (8),

$$|X_i| = \lceil \tfrac{1}{2}n \rceil + i, \qquad |Y_i| = \lfloor \tfrac{1}{2}n \rfloor - i. \tag{10}$$

We prove by induction on i that

$$\delta(X_i) \geq \lceil \tfrac{1}{2}\delta \rceil + i, \tag{11}$$

for $i \in \{0, 1, \ldots, \lfloor \frac{1}{2}\delta \rfloor\}$.

When $i = 0$, (11) holds by (i) and the choice of X. We may assume $\delta \geq 2$ or the argument is complete. Now suppose (11) is true for some $i \in \{0, 1, \ldots, \lfloor \frac{1}{2}\delta \rfloor - 1\}$. Then, from (10) and (11),

$$\delta(Y_i) \leq \lfloor \tfrac{1}{2}\delta \rfloor - (i + 1). \tag{12}$$

Therefore, from (9) and (12)

$$\delta(X_{i+1}) \geq \lceil \tfrac{1}{2}\delta \rceil + (i + 1). \tag{13}$$

By induction (11) is true for each $i \in \{0, 1, 2, \ldots, \lfloor \frac{1}{2}\delta \rfloor\}$. In particular (11) is true for $i = \lfloor \frac{1}{2}\delta \rfloor$. Hence from (10) and (11), $G \in \mathscr{C}(n, \delta, \lfloor \frac{1}{2}\delta \rfloor) \subseteq \mathscr{C}(n, \delta)$ and the contradiction proves the theorem in this case.

Case 2. From Case 1 we may assume that $\delta(X) < \lceil \frac{1}{2}\delta \rceil$. Suppose $i \in \{0, 1, \ldots, \lfloor \frac{1}{2}\delta \rfloor\}$. A partition (X_i, Y_i) of $V(G)$ is defined recursively as follows:

(i) $Y_0 = X$, $X_0 = V(G) \backslash Y_0$,

(ii) using Lemma (ii), choose (X_{i+1}, Y_{i+1}) so that

$$Y_{i+1} \in \max(|Y_i| - 1, G), \qquad X_{i+1} = V(G) \backslash Y_{i+1} \tag{14}$$

and

$$\delta(X_{i+1}) \geq \delta - \delta(Y_i), \tag{15}$$

From the definition of X, (i) and (14),

$$|X_i| = \lfloor \tfrac{1}{2}n \rfloor + i, \qquad |Y_i| = \lceil \tfrac{1}{2}n \rceil - i. \tag{16}$$

Now repeat the argument of Case 1. $\square$

3. Edge density and graph decomposition

In this section we will always assume that n and δ are integers and that $n \equiv \delta \pmod 2$. Write

$$f(n, \delta) = \tfrac{1}{8}(n^2 + (6\delta - 10)n - 3\delta^2 + 6\delta + 8).$$

Conjecture 1. *Let G be a graph with n vertices. Suppose $n-2 \geq \delta \geq 1$. Then*
(1) *if $n \neq \delta+4$ and $q(G) \geq f(n, \delta)$, $G \in \omega(n, \delta)$,*
(2) *if $n = \delta+4$ and $q(G) \geq f(n, \delta)+1$, $G \in \omega(n, \delta)$.*

Remarks

(4) The graph $G = K_{\delta-1} + (\bar{K}_{\lfloor \frac{1}{2}n \rfloor - \lfloor \frac{1}{2}\delta \rfloor + 1} \cup K_{\lceil \frac{1}{2}n \rceil - \lceil \frac{1}{2}\delta \rceil})$ has $f(n, \delta)-1$ edges and $G \notin \omega(n, \delta)$. Except in the case when $n = \delta+4$ this graph is edge maximal with respect to $G \notin \omega(n, \delta)$. When $n = \delta+4$, G is not edge maximal but the addition of any one edge e to the induced subgraph isomorphic to $\bar{K}_{\lfloor \frac{1}{2}n \rfloor - \lfloor \frac{1}{2}\delta \rfloor + 1}$ of G yields the maximal subgraph $G+e$. Hence, in this respect, if the conjecture is true then it is sharp.

(5) A weaker form of Conjecture 1 is given by

Conjecture 2. *Let G be a graph with n vertices. Suppose $n-2 \geq \delta \geq 1$ and $n \neq \delta+4$. Suppose $\delta(G) \geq \delta-1$ and $q(G) \geq f(n, \delta)$. Then $G \in \omega(n, \delta)$.*

It may be that Conjecture 2 is helpful but our own experience in the case $\delta = 2$ suggests otherwise.

(6) We prove rather easily:

Theorem 3. *Conjecture 1 is true when $\delta = 1$ and when $n \in \{\delta+2, \delta+4\}$.*

Proof. The argument is a simple exercise. □

(7) We have "proved" (the quotations are because our proof is so long and complicated that there must be a high probability that some of the details are incomplete) that Conjecture 1 is true when $\delta = 2$. To be precise we have proved:

Theorem 4. *Let n be an integer such that $n \neq 3$. Let G be a graph with $2n$ vertices and $q(G) \geq \frac{1}{2}(n^2+n+2)$. Then $G \in \omega(2n, 2)$.*

Proof. As previously mentioned the "proof" is very complicated. However an outline of the method of proof is given in [15]. □

Some of the difficulty in proving Theorem 4 may be avoidable if the following stronger result is true.

Conjecture 3. *Let n be an integer such that $n \neq 3$. Let G be a graph with $2n$ vertices and $q(G) \geq \frac{1}{2}(n^2+n+4)$. Then $G \in \omega(2n, 2, 1)$.*

Of course Conjecture 3 is rather wild and a more sensible conjecture might be to replace the edge condition by $q(G) \geq \frac{1}{2}(n^2+n+4)+\lambda$, where λ is a constant independent of n. If Conjecture 3 is true then it is sharp in the sense that the graph G consisting of disjoint copies of K_n and a cycle C_n together with one edge joining a vertex of K_n to a vertex of C_n has $p(G)=2n$, $q(G)=\frac{1}{2}(n^2+n+2)$ and $G \notin \omega(2n, 2, 1)$.

(8) The reader may be slightly puzzled as to the origin of Conjecture 1. In fact it is really related to a problem in Ramsey Theory (see [4, Conjecture 2]). We use a notation of [4]. Suppose k and δ are integers such that $k \geq 3$ and $\delta \geq 1$. Let H be a graph on n vertices such that $p(H)=n=2k+\delta$ and $q(H) \leq \frac{1}{2}(3k^2+3k-2)$. Write $G=\bar{H}$. Then $p(G)=n$, $n \equiv \delta \pmod 2$ and $q(G) \geq \frac{1}{8}(n^2+(6\delta-10)n-3\delta^2+6\delta+8)$. Thus, if Conjecture 1 is true, $G \in \omega(n, \delta)$. Hence there exists $i \in \{0, 1, \ldots, \lfloor\frac{1}{2}\delta\rfloor\}$ such that (X_i, Y_i) is a weak (i, δ)-partition of $V(G)$. Now in H colour all the edges of $\langle X_i\rangle \cup \langle Y_i\rangle$ red and all the other edges of H blue. From the definition of an (i, δ)-partition this colouring of H does not contain a red $K_{1,\frac{1}{2}(n-\delta)}$ (where $k=\frac{1}{2}(n-\delta)$ or a blue K_3, i.e., $H \not\to (K_{1,k}, K_3)$. Hence the size Ramsey number $r_q(K_{1,k}, K_3)$ satisfies

$$r_q(K_{1,k}, K_3) \geq \binom{2k+1}{2} - \binom{k}{2},$$

and therefore, from Theorem 1 of [4], this is an equality. Hence if Conjecture 1 is true then Conjecture 2 of [4] is also true. Theorem 4 answers the particular case of Conjecture 2 in [4] when $\delta = 2$.

We may also restate Theorem 1 in this context. Suppose H is as above except that now we do *not* impose any bound on the number $q(H)$ of edges on H. Suppose $\Delta(H) \leq 2k-1$. Then $\delta(G) \geq \delta$. Hence, by Theorem 1, using the same colouring as above, $H \not\to (K_{1,k}, K_3)$. This statement is best possible as may be seen by putting $H = K_{2k+1}$ and applying Turan's theorem. This was pointed out to me by a referee.

4. Generalizations

Theorem 5 below is a natural and easy generalization of Theorem 1. Firstly we need to generalize the concept of an (i, δ)-partition.

Suppose r is a rational number such that $1 \geq r \geq 0$. Write $r^* = 1-r$. Suppose i and δ are integers with $n-1 \geq \delta \geq 0$ and $\lfloor r^*\delta\rfloor \geq i \geq 0$. A partition (X, Y) of $V(G)$ is an (i, r, δ)-*partition* if

(1) $|X| = \lceil rn\rceil + i$, $|Y| = \lfloor r^*n\rfloor - i$,

(2) $\delta(X) \geq \lceil r\delta\rceil + i$, $\delta(Y) \geq \lfloor r^*\delta\rfloor - i$,

(3) X or Y is dense.

Let

$$\mathscr{C}^*(n, \delta, r, i) = \{G: |V(G)| = n,\ G \text{ contains an } (i, r, \delta)\text{-partition}\}$$

and

$$\mathscr{C}^*(n, \delta, r) = \bigcup_{i=0}^{m} \mathscr{C}^*(n, \delta, r, i), \quad \text{where } m = \lfloor r^*\delta\rfloor.$$

Theorem 5. *Suppose r is a rational number such that $\frac{1}{2} \leq r \leq 1$. Suppose $r = s/t$, where $(s, t) = 1$. Let G be a graph with n vertices. Suppose δ is an integer such that $n - 1 \geq \delta \geq 0$ and such that $n \equiv \delta \bmod(t)$. Suppose $\delta(G) \geq \delta$. Then $G \in \mathscr{C}^*(n, \delta, r) \cup \mathscr{C}^*(n, \delta, r^*)$.*

A natural strengthening (in the spirit of [11, Theorem 1]) of Theorem 1 is given in Conjecture 5 below. Let $\mathscr{K}(G)$ denote the connectivity of G. Suppose $m \in \{0, 1, \ldots, \lfloor\frac{1}{2}\delta\rfloor\}$. Write

$$\omega_m(n, \delta) = \bigcup_{i=0}^{m} \omega(n, \delta, i).$$

Conjecture 5. *Let G be a graph with n vertices. Suppose δ is an integer such that $n - 1 \geq 2\delta \geq 0$, and suppose $n \equiv \delta \pmod 2$. Suppose $\delta(G) \geq \delta$ and $\mathscr{K}(G) \geq \mathscr{K}$. Then $G \in \omega_m(n, \delta)$, where $m = \max\{0, \lfloor\frac{1}{2}\delta\rfloor - \mathscr{K}\}$.*

Remarks

(9) When $\mathscr{K} = 0$ Conjecture 5 follows from Theorem 1 for $n \equiv \delta \pmod 2$. When $\mathscr{K} = 1$ and $\delta = 2$ then Conjecture 5 follows from [14, Theorem 2.1]. In this context see also [5] and [9]. Recently we have proved this conjecture for $\mathscr{K} = 1$ and all δ. We rely heavily on Theorem 1. The proof is not entirely trivial.

(10) In Conjecture 5 the condition $n - 1 \geq 2\delta \geq 0$ is needed. For example let G be the complement of the Petersen graph. Assume that $\delta(G) = \delta = 6$. It is easy to check that G contains a weak $(1, \delta)$-partition but no weak $(0, \delta)$-partition.

(11) In the spirit of Conjecture 5 it is natural again to speculate that Theorem 4 might generalize.

Theorem 6. *Let n be an integer such that $n \neq 3$. Let G be a graph with $2n$ vertices and $q(G) \geq \frac{1}{2}(n^2 + n + 2)$. Then if G is connected, $G \in \omega_0(2n, 2) = \omega(2n, 2, 0)$.*

Using the techniques of [14] the proof of Theorem 6 is quite simple although the details are too long and complicated to give here.

(12) Catlin recently informed me that Theorem 1, with $\mathscr{C}(n, \delta)$ replaced by $\omega(n, \delta)$ can be proved using a result of Lovász. Let G^c denote the complement of G and let $\Delta(G)$ denote the maximum degree of G. Write $\Delta(X^c)$ for $\Delta(\langle X\rangle_{G^c})$ $(X \subseteq V(G))$. Suppose $n \equiv \delta \pmod 2$. Then G has a *weak* (i, δ)-partition if and only if there exists a partition (X, Y) of $V(G^c)$ such that: (i) $|X| - |Y| = 2i$; (ii) $\Delta(X^c) \leq \frac{1}{2}(n - \delta - 2)$, $\Delta(Y^c) \leq \frac{1}{2}(n - \delta - 2)$.

Lovász [10] proved the following. Let s, t be integers such that $s + t = \Delta(G) - 1$. Then there exists a partition (X, Y) of $V(G)$ with $\Delta(X) \leq s$ and $\Delta(Y) \leq t$. Now write $s = t = \frac{1}{2}(n - \delta - 2)$ and, considering G^c rather than G, we obtain the weaker version of Theorem 1.

Theorem 1 of this paper is stronger and possibly its proof is of independent interest. The knowledge that there exists a "strong" (i, δ)-partition (X, Y) is vital in verifying Conjecture 5 for the case $\mathscr{K} = 1$. In particular the fact that either X or Y is dense is crucial in the proof.

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